Teaching mathematics with reasoning tools: learning, teaching and curriculum planning

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Where am I from …ResearchLab: IT and learning design

- Research themes
  - Students as producers and designers
  - Game Based Learning
  - ICT and core topics in primary and secondary school
  - ICT and tertiary education
  - Organizational aspects of ICT and learning

The title?

- Teaching mathematics with reasoning tools: learning, teaching and curriculum planning

- Reasoning tools := CAS, DGS, Tools for specific tasks
- Learning, Teaching, Curriculum planning:=
  - What kind of **learning** processes and leaning difficulties can we observe
  - What kind of values about our discipline are we trying to **teach** and what conflicts are we experiencing as teachers
  - What topics should be in the **curriculum** and how do we organize curriculum materials such a textbooks

Overview of talk

- Explain about what I do in order to convey from where I speak
- The situation in Denmark
- Three examples of reasoning tool problems
  - instrumental understanding and learning difficulties
  - Triangles and trigonometry
  - Proofs and CAS
- Zoom out, discussion and conclusions

My research portefolio

- I am your friend (seriously)
  – But I will try to challenge you here.
- Research in mathematical practice
- Young students creative mathematical ICT work
- Teachers use of technology
- … and the influence of CAS/DGS/ect on secondary education.

From where I speak 1: Mathematical Practices

- Philosophical and educational project
- Objective:
  - Gain empirical knowledge about mathematical research practice
    - Problem solving, posing and choosing
    - the use of materiality/representation in mathematical activities
  - Develop knowledge that informs mathematics education
- Mean:
  - Interview researchers of mathematics about work practices, how they attack and solve mathematical problems, how they use visual representations and computers, and how they collaborate with peers.

Misfeldt, M., & Johansen, M. W. (2015). Research mathematicians' practices in selecting mathematical problems. Educational Studies in Mathematics, 89(3), 357-373. DOI: 10.1007/s10649-015-9605-3

Working with mathematical problems

- *"Try to describe how you start to work on a mathematical problem."*
  - *Well, in fact I believe it all starts a bit earlier, because you also need to find the problem you wish to attack. There are so many problems, so you have to make some kind of selection.*
  - *Well, the most important is of course … The most essential, which comes earlier, is that you have to ask some questions, right, and then you can ask where those questions come from.*
- "How do you chose what problem to work on":
  - "As a matter of fact, I believe… in fact, it is the hardest part."
  - *"It is not easy. The great art in mathematics is to find a problem that you have ideas about how to solve.*
And in fact have a chance of solving. […] The greatest art is not to solve problems, the greatest art is to find the problems we can solve."

Mathematical problems

- Is a context for mathematical investigation
- Leads to new mathematical problems
- Is not a thing to be solved and left
- **The choice** of what problems to work on is crucial for a mathematician, he/she balances:
  - Interest
  - Head start
  - Audience (community)

From where I speak 2: Creative Digital Mathematics

- Teachers as designers of learning environments/ scenarios
- Students as producers of board games and visual layouts with GeoGebra.
- 2011-2014 (approx 70 teachers total)
- Goal was to push
  - The typical student activity in primary math ed. towards producing and exploring with and within mathematics
  - Teacher collaboration and teacher roles, towards producing and exploring new learning situations in collaboration.

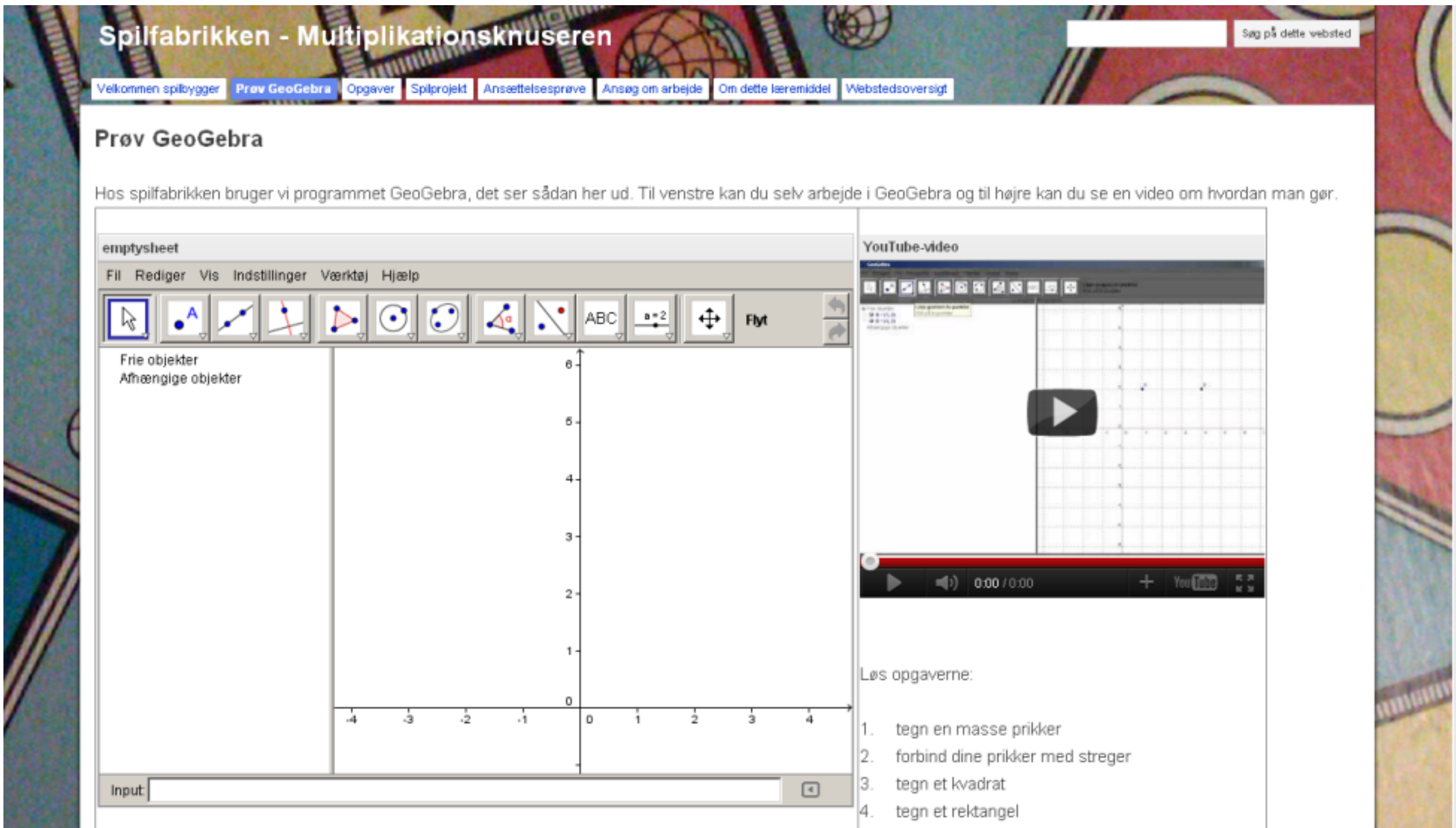
Misfeldt, M., & Zacho, L. (2016). Supporting primary-level mathematics teachers' collaboration in designing and using technology-based scenarios. Journal of Mathematics Teacher Education, 19(2-3), 227-241. DOI: 10.1007/s10857-015-9336-5

Teachers as scenario designers

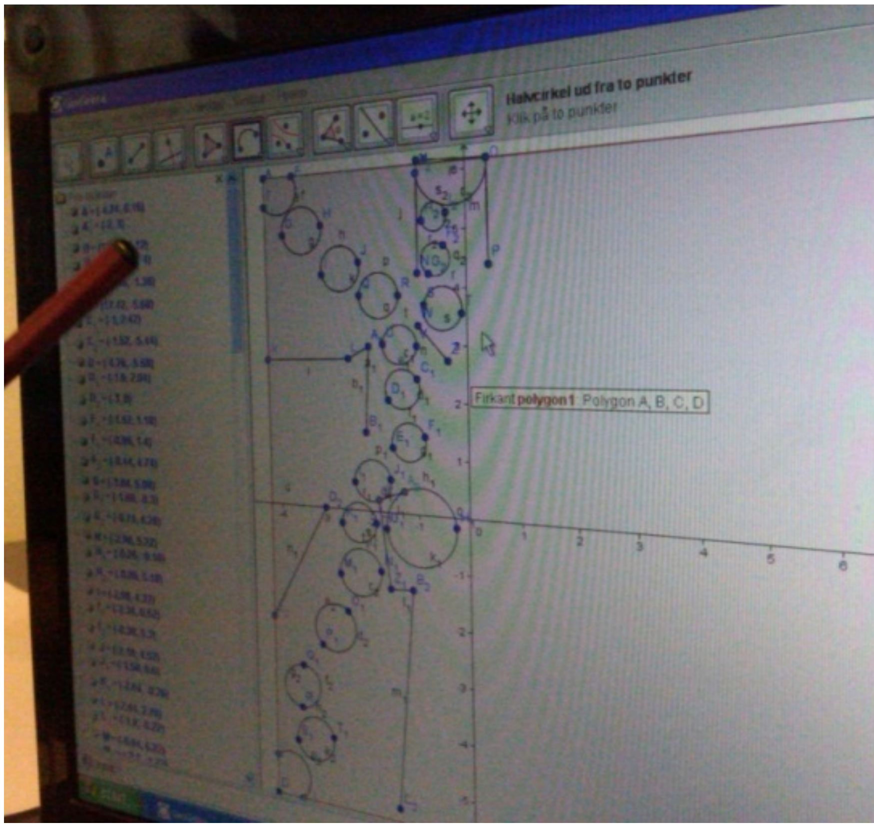
Currently 36 scenarios on the webpage

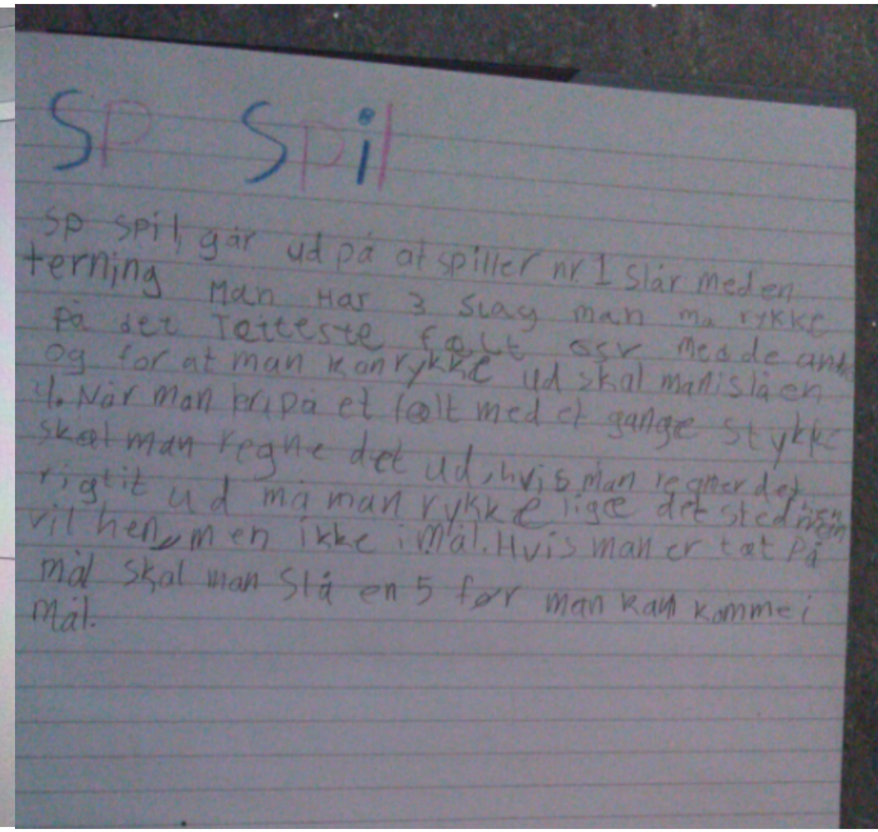
A prototypical scenario – the game factory

1.Welcome game builder 2. Try GeoGebra 3. Tasks 4. Game project 5. Test 6. Apply for work



Example of game





The purpose of SP Game is that player number one throws a dice three times, and if you throw a “4“ you can move to the nearest next field. When get to a field with a multiplication calculation, you should do the calculation right, and then you can move all over the game board but not into the target zone. If you are next to the target zone you have to throw a 5 with the dice in order to get into the target.

Sorry for the long intro

- The goal of my talk is to zoom into some of the problems that I see with reasoning tools in secondary education in Denmark
- The anecdotes and rumors
  - All student lack skills in mathematics (and it is all because of CAS!☺)
  - Clever students are bored in mathematics
  - New types of learning difficulties emerges
- The facts
  - Prominent math education scholars argue strongly against to much use of CAS and similar tools in Danish upper secondary
  - Teachers disagree about the value of these tools
  - Policy document promote ICT

The State of the: to use tools or not to use tool

- Tools are an essential part of both doing mathematics and teaching mathematics
- The use of efficient cognitive tools characterize mathematics
- Tools influences mathematical activities outside school
- And CAS works – a meta analysis from 2008 shows it is efficient (Tokpah 2008)

- I am not here to advocate against the use of tools
  - But we do need at deeper understanding of the problems that comes from using these tools

Tokpah, C. (2008). *The Effects of Computer Algebra Systems on Students' Achievement in Mathematics.* (Electronic Thesis or Dissertation). Retrieved from https://etd.ohiolink.edu/

The State of The Art: main concerns

- Understanding tools and technologies in mathematics education is red ocean
- The history
  - From change in discipline to understanding new learning processes, to teaching situations
- The practical aspects
  - Cabri, GeoGebra, Geomatech …
- The theoretical aspects
  - Understanding humans with tools, blackboxing, instrumental genesis, cognitive artifacts
- The core problems
  - Assessment, teaching with tools
- And the values
  - Is CASmath as good as chalkmath … what about computational thinking.

My approach today

- Examples of tensions and problems
- Goal: friendly critique
- Levels in understanding the situation
  - Learning and reasoning tools
  - Teaching and reasoning tools
  - Curriculum design and reasoning tools
- Three examples
  - CAS and algebraic work
  - Triangles and trigonometry
  - CAS and proofs

The Danish situation

Successful implementation – problems and critique

Technology in lower and upper secondary mathematics education in Denmark

- Lower secondary education
  - major leap in technology use in the last 2-3 years
  - Main technologies are GeoGebra and Excel
  - A lot of CAI tools are used
  - Part of the curriculum from 2015
  - Seems successful – but hard to get all teachers onboard
- Upper secondary
  - At least ten years of heavy technology use
  - CAS systems (maple, TI inspire, mathcad) are dominant
  - Part of the curriculum from 2005
  - Intentions is that CAS should not only serve the role of a tool for solving problems, etc. but also be seen as an *instrument for conceptual understanding*.
  - The details of the implementation is left to the schools, the teachers, and the textbook authors. And it is rather diverse.

The Danish discussion

Pro or con ict

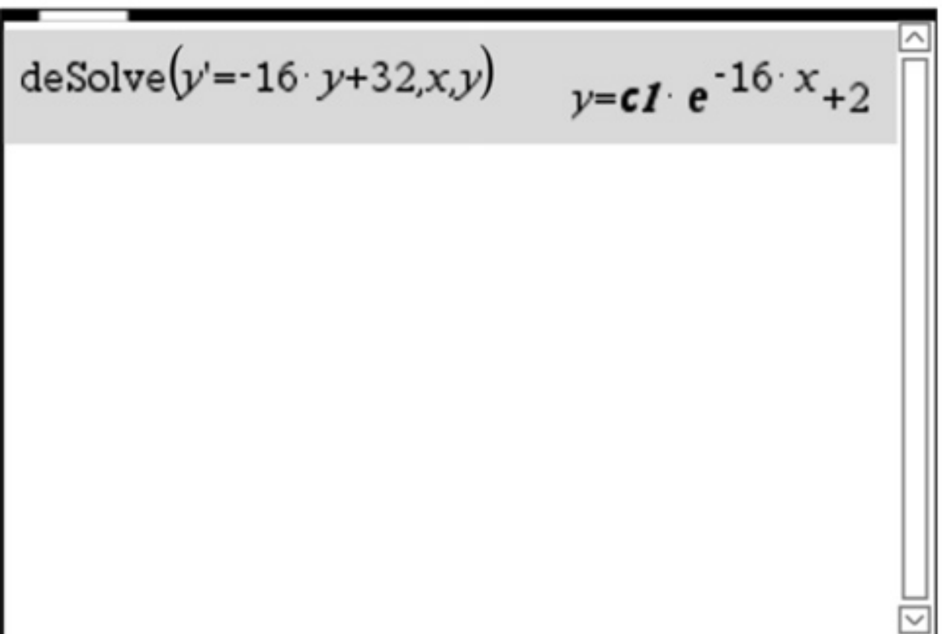
- *IT-alistic* versus *mathematical* approach to mathematics education (M. Niss in various talks and interviews)
  - We might not agree – but we need to take notice when very strong scholars in the field point to problems related to technology use
- Informal reports on problems in upper secondary, but definitely not consistent.
- Critique of the use of CAI tools in lower secondary/primary

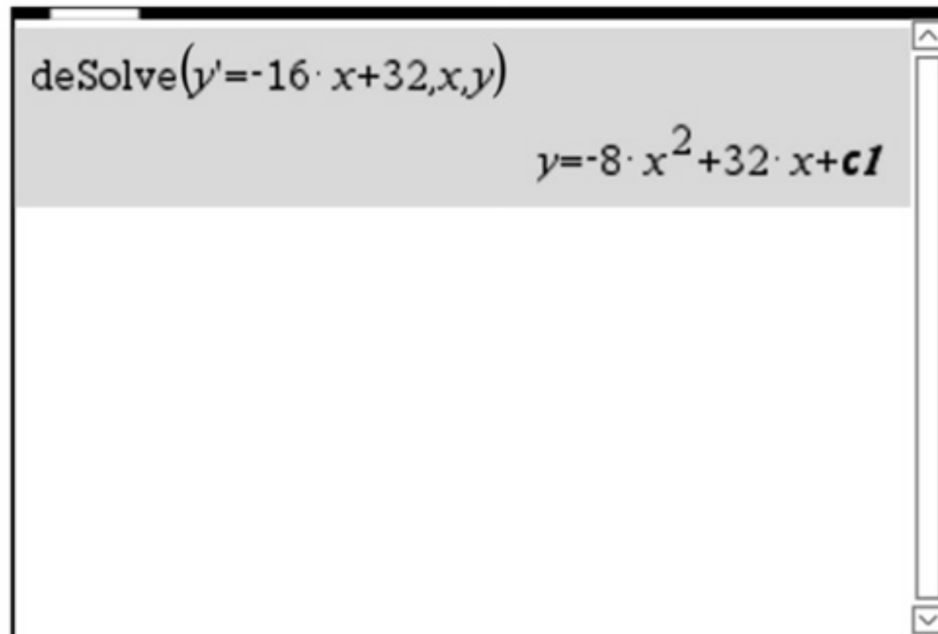
Learning difficulties

Induced by CAS use

A simple task and an innocent mistanke

- Given dN /dt = –16N + 32 with the boundary condition that N (10) = 1, find an expression for N (t)
- Case study based on interviews, documents and observation





Jankvist, U., & Misfeldt, M. (2015). CAS-Induced Difficulties in Learning Mathematics? For the Learning of Mathematics, 35(1), 15-20.

Problem

- We see a large number of such very basic mistakes in Danish upper secondary school
- Mistakes that we can understand in terms of lack of *relational understanding* (Skemp 1976) as well as in terms of the development of instrumented techniques that changes the landscape of mathematical relations.
- Example of instrumented techniques
  - See an equation -> solve("the equation", x)
  - See a differential equation -> desolve("the D-equation, x, y)

The viscious circle of reification (Sfard 1991)

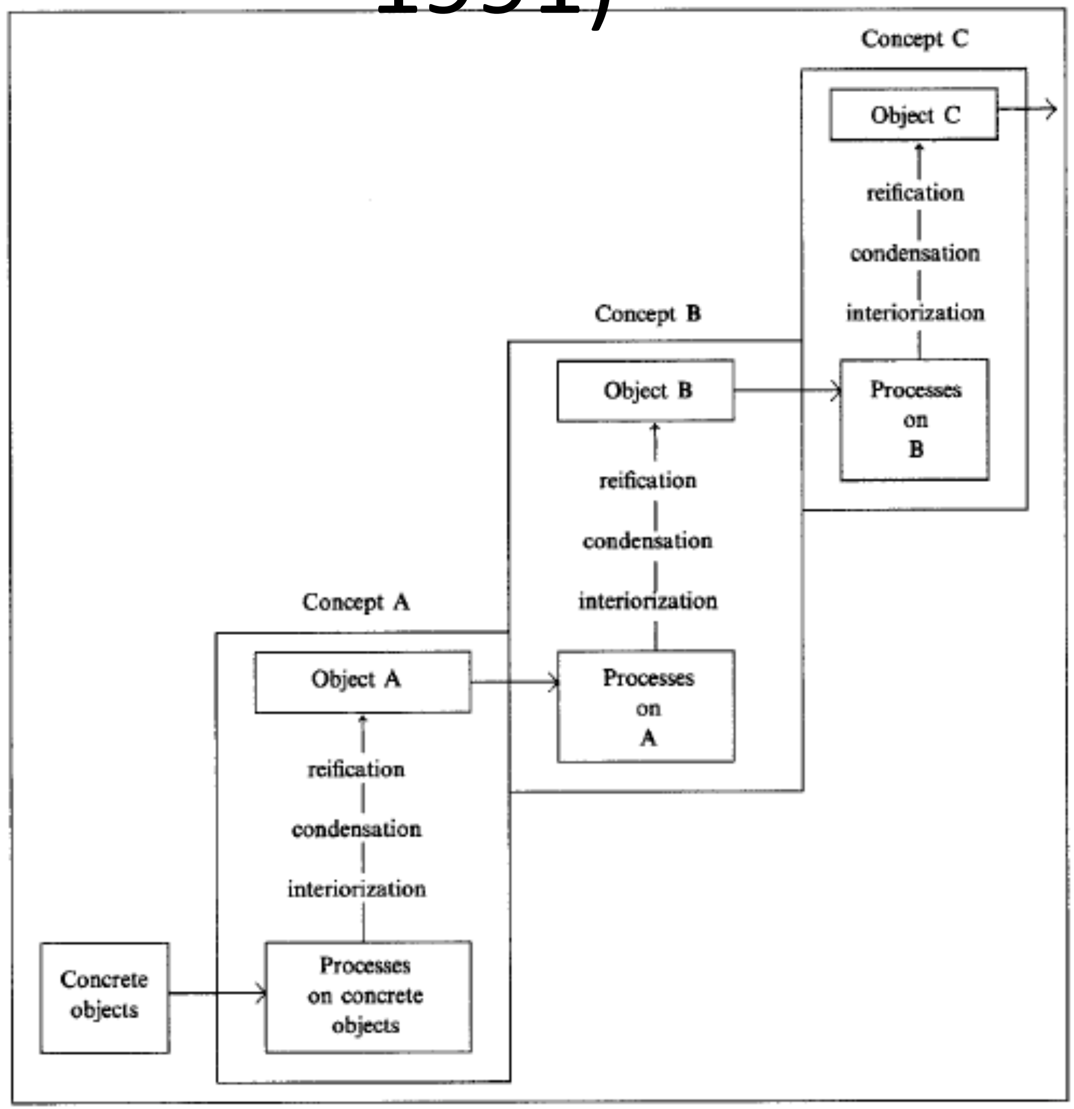


Fig. 4. General model of concept formation.

CAS-Related learning difficoulties in mathematics

- Procedural understanding
- Students can act without of relational understanding and still obtain requested results
- In that sense CAS can help ”overcome” Sfards Vicious circle which then becomes a vicious spiral and instead of keeping students away from performing mathematically – it allows them to perform in a completely superficial way.

Teaching trigonometry

Salient influence of technology

Trigonometry in secondary school in Denmark

- Lower secondary
  - Simple triangle problems (since 2009)
- Upper secondary
  - Complex triangle problems (including the various trigonometric reations)
  - The unit circle
  - The trigonometric functions (the later years/ higher levels)

Misfeldt, M. (2014). Trekantsberegninger og teknologi: et eksempel på hvordan teknologi har (eller bør have) indflydelse på udvikling af Matematikcurriculum . Mona, 2014(1), 27-43.

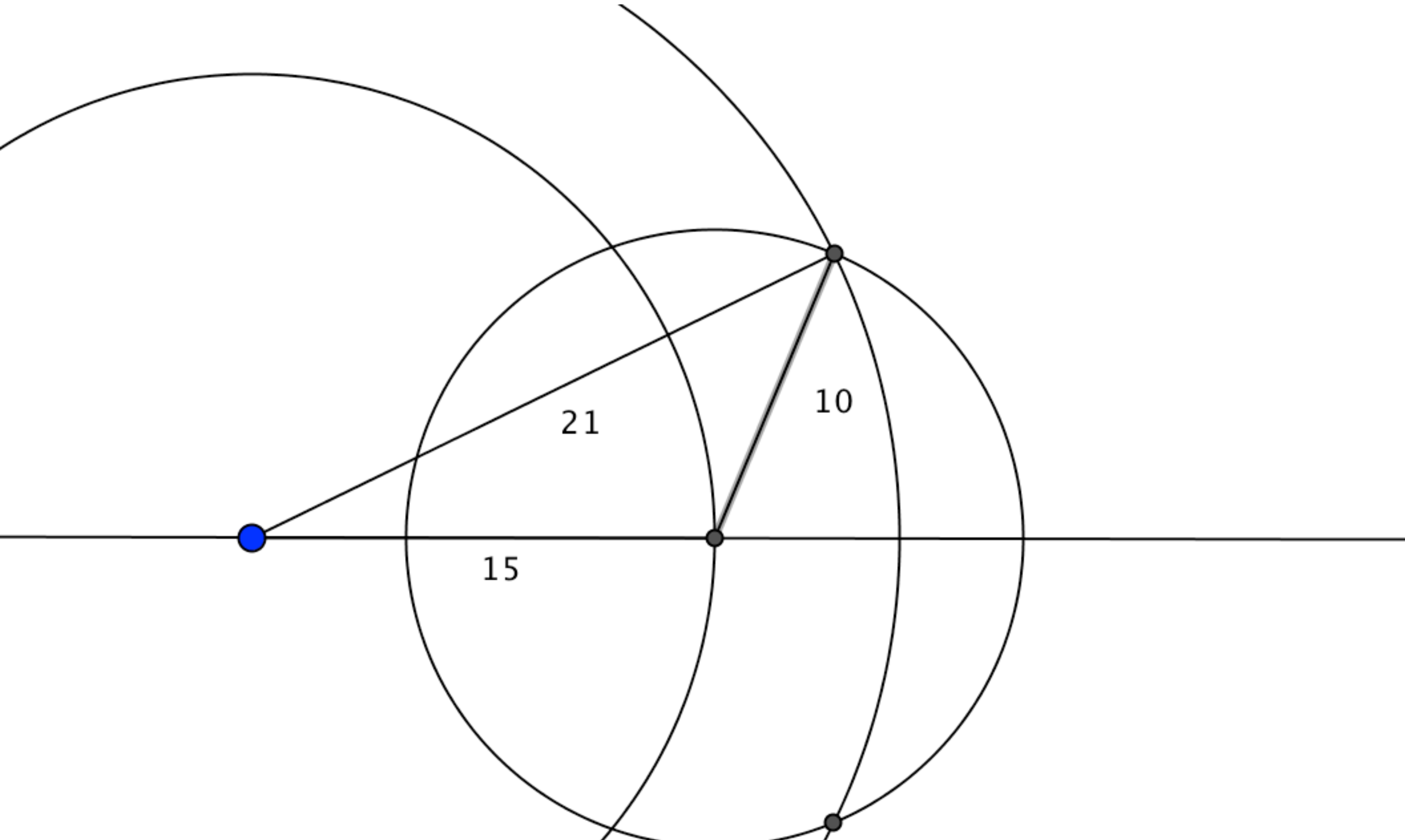
The technological situation

- DGS with precise measuring tools (gives students the ability to create “exact” drawings
- Triangle calculators automates the algebraic/ trigonometric solution strategy

Example

- ”In a triagle ABC the sides are a=10, b=15 og c=21, find the angle A”

Euclidian – on steroids



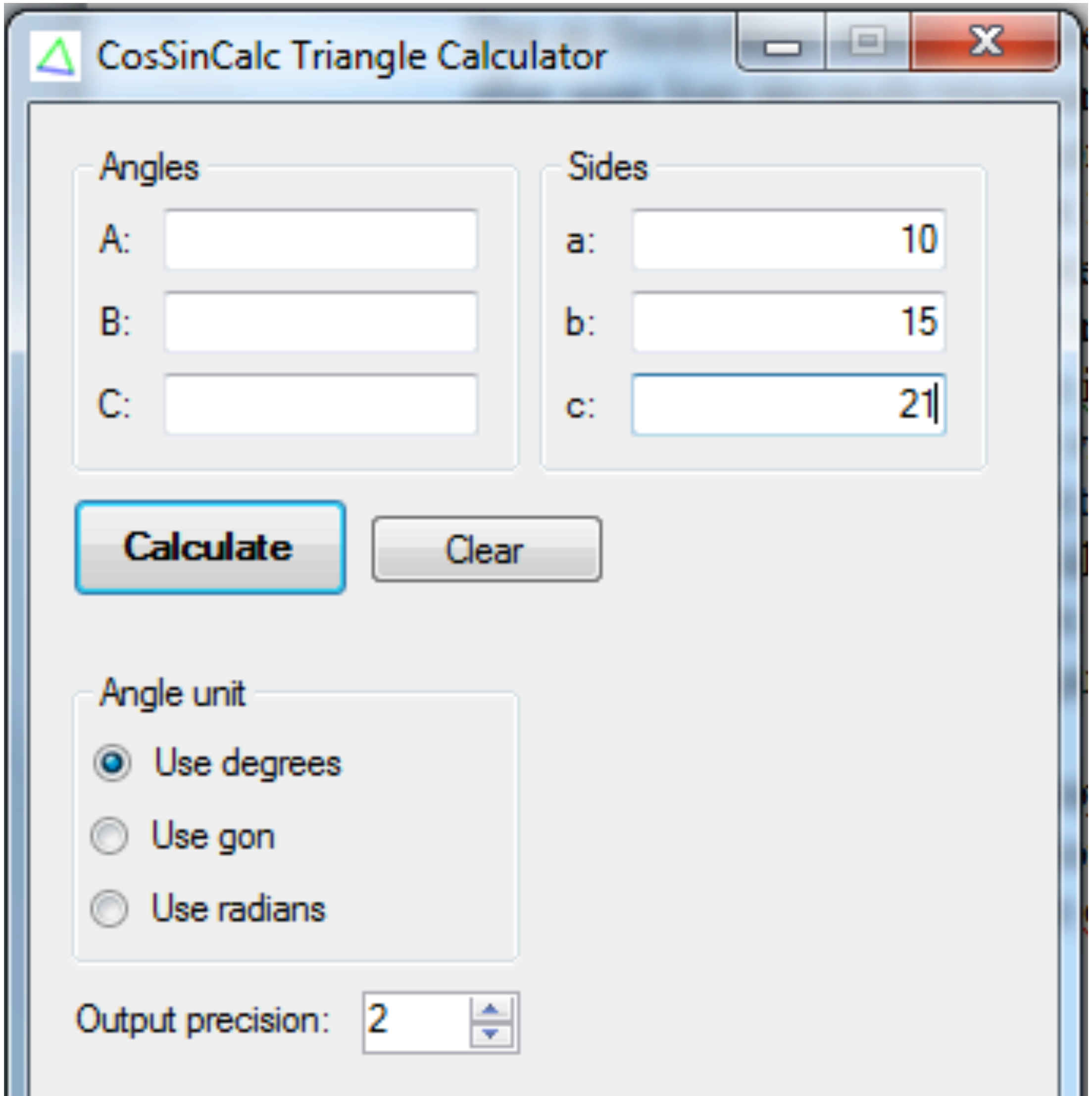
Algebraic

$$a^2 = b^2 + c^2 - 2 \cdot b \cdot c \cdot \cos A$$

$$A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2 \cdot b \cdot c}\right)$$

$$= \cos^{-1}\left(\frac{15.00^2 + 21.00^2 - 10.00^2}{2 \cdot 15.00 \cdot 21.00}\right) = 26.05^{\circ}$$

Automatic


CosSinCalc Triangle Calculator
Angles
A:
B:
C:
Sides
a: 10
b: 15
c: 21
Calculate
Clear
Angle unit
Use degrees
Use gon
Use radians
Output precision: 2

Looking at the situation

- We have three strategies towards the problem
  - Euclidian
  - Algebraic
  - Automatic
- Two of these are “new”
- This changes the dynamic of teaching trigonometry
  - cos, sin and tan are obsolete for dealing with triangle situations
  - There is a strategy for students to avoid working with trigonometry (automated solution)

Influence on curriculum level

- Triangle trigonometry looses relevance
- Still relevant as mean to teach about the trigonometric functions
- But is currently taught 3 years before the trigonometric functions

- This situation is not unusual
  - Long division
  - Analyzing a function (find zeros, f'(x)=0 .. After a lot of algebraic work you can draw the graph)

On the pedagogical level

- How do we answer student who always take the easy and automated approach?
- What do we say to students that asks “why do we need to learn the trigonometric approach”
  - It is more correct
  - It prepares you for future study
  - It is pretty and historically interesting
- What image of math do we convey by answering like this
  - Authoritarian
  - Inefficient
  - Text oriented
  - Preparation to life – not life itself

Proofs with computer algebra systems

Implementing reasoning tools in textbooks

CAS-Proofs

- As described CAS is massively present in upper secondary mathematics in Denmark (since a 2005 reform)
- A notion and practice of “CAS proofs” or “CAS-assisted proofs“ have emerged in the textbooks.
- A CAS assisted proof is a textbook proof of a statement or theorem, where one or several steps are performed by a CAS tool.

What kind of CAS Proofs do we see in tetbooks

- Almost only CAS
- CAS for a specific aspect
- Checking Proofs with CAS

- There might be other forms

Example 1: Complete outsourcing

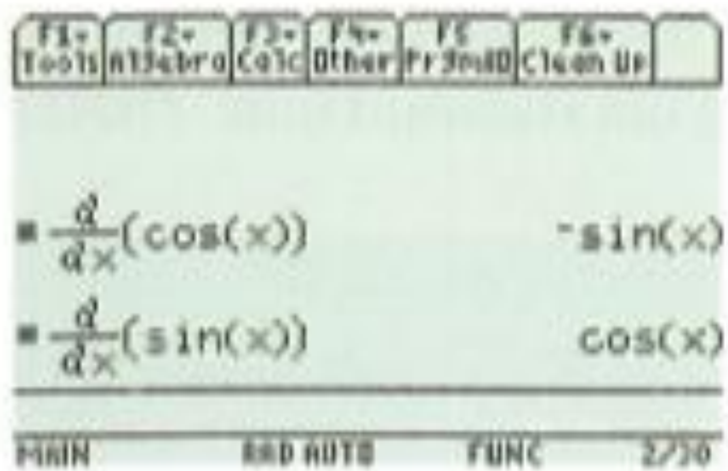
Differentialkvotienten af cos og sin

Sætning 1

Funktionerne cosinus og sinus er differentiable i ethvert reelt tal x, og

$$\cos'(x) = -\sin(x)$$
$$\sin'(x) = \cos(x).$$

Vi giver et CAS-bevis, jf. figur 109.



Figur 109

Techno authoritarian proof scheme

- No attempt is made at explaining or visualizing why cos'(x) = -sin(x) and sin'(x) = cos(x)
- The "CAS proof" is both convincing and trustworthy if we trust the CAS, but is does not explain it is only a proof that convince (Hanna 1989),
- and it refers to as external technological authority (the CAS), that can be manipulated by the student! Leading to instrumental genesis (if I don't know the truth value – I ask CAS)
- "techno-authoritarian" external conviction proof scheme.

Example 2: Outsourcing and checking of specific technical aspects

Proving that the derivative of sin(x) is con(x) by algebraic manipulations and the limit.

SÆTNING 1
Den afledede af $f(x) = \sin x$ er $f'(x) = \cos x$.

Bevis. Differenskvotienten er

$$\frac{\Delta y}{h} = \frac{f(x+h)-f(x)}{h} = \frac{\sin(x+h)-\sin x}{h}\ .$$

Man kan vise, at der gælder en trigonometrisk formel for en differens mellem to sinus-værdier:

$$\sin x - \sin y = 2\cdot\cos\frac{x+y}{2}\cdot\sin\frac{x-y}{2}\ ,$$

og benytter vi denne formel, fås

$$\frac{\Delta y}{h} = \frac{2\cdot\cos\frac{2x+h}{2}\cdot\sin\frac{h}{2}}{h} = \cos(x+\tfrac{h}{2})\cdot\frac{2\cdot\sin\frac{h}{2}}{h} = \cos(x+\tfrac{h}{2})\cdot\frac{\sin\frac{h}{2}}{\frac{h}{2}}\ . \qquad (1)$$

Nu skal vi lade h nærme sig 0. Da cos er kontinuert, gælder

$$\cos(x+\tfrac{h}{2}) \to \cos x \ \text{ for } \ h \to 0\ . \qquad (2)$$

All is done algebraically (by hand) except the limit sin(k)/k when k approaches 0, which is handled in three different ways, two of them involving CAS.

1. By a visual argument using the unit circle
2. By experimenting with small numbers for k
3. By asking the CAS to calculate

Vi skal derfor bestemme grænseværdien af $\frac{\sin\frac{h}{2}}{\frac{h}{2}}$ når h nærmer sig 0.

Da $\frac{h}{2}$ nærmer sig 0, når h gør det, kan vi også sige, at vi skal finde

$$\lim_{k\to 0}\frac{\sin k}{k}.$$

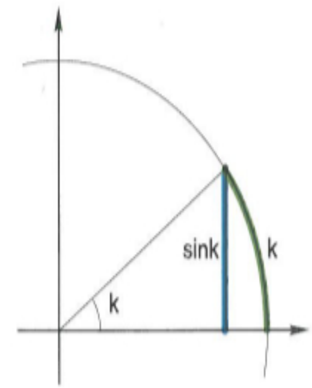
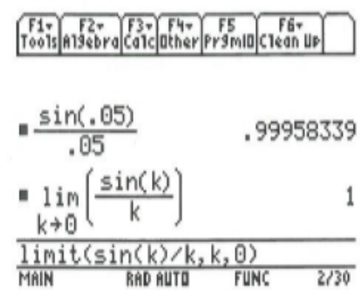


Fig. 9

På enhedscirklen kan vi se (fig. 9), at hvis k er et tal (vinkel i radianer) tæt på 0, er sin$k$ (y-koordinaten til retningspunktet for k) og k (buelængden for radiantallet k) næsten lige store, så vi har, at

$$\frac{\sin k}{k}\to 1 \text{ for } k\to 0.$$



Dermed får vi af (1) og (2), at

$$\frac{\Delta y}{h}\to \cos x\cdot 1=\cos x \text{ for } h\to 0,$$

og da denne grænseværdi er differentialkvotienten for $f(x)=\sin x$, er

Augmenting and checking the proof by use of CAS

- We have almost a full algebraic proof, and CAS augments in two ways
  - The use of CAS for calculation of examples with very small values of *k* is experimental and does not play a formal role in the proof.
  - The use of CAS to verify the limit calculation play the role of checking an algebraic and visual argument

Questions about CAS and that should be addressed

- *Does the CAS use establish truth?*
- *Does the CAS use allow interaction and experimentation*?
- Is the argumentation *inductive, deductive or authoritarian*?
- Does the argument highlight *important aspect of the proof* or the mathematical relationships?

Misfeldt, M., & Thomas Jankvist, U. (2016). Instrumental genesis and proof: understanding the roles of computer algebra systems in textbook proofs. Paper presented at ICME, 13 International Congress on Mathematics Education, Hamburg, Germany.

Recab

- Reasoning tools can remove students grounding and create a new type of learning difficulties
- Change in technological situation is a change in the learning ecology around specific mathematical topics
- Tools affect the formats/genres that are used in mathematic situation, leading to complex/ paradoxical phenomenon's such as automated solutions and distributed proofs

The Levels

- Learning and reasoning tools
  - We do have strong theoretical tools e.g. instrumented techniques but I would argue for mixing with other theories about learning mathematics.
- Teaching and reasoning tools
  - How should teachers react to scripts such as the automated triangle calculators, or heavy CAS use in algebraic exercises
- Curriculum design and reasoning tools
  - When to teach what
  - How to refer to tools in textbooks

Possible actions (1): On the level of school systems

- Regulate tools (in teaching, in exams)
- In service training and discussion among teachers (norms and practices differs greatly)
- Continuously consider the technological situation and change syllabus accordingly

Possible actions (2): On the level of research and math education community (last slide…)

- Focus on these paradoxes – unfold and understand them – I would ague for a battery of problem situations
- Activate learning theories and theories about the teaching and learning of mathematics
  - Not to prescribe – but to understand
- We are mathematics!
  - We – math educators – should seriously consider what kind of image of the discipline we convey to student
    - Pushing buttons, learning old redundant strategies to problem solving, and distributing mathematical insights to machines is not what we want students to learn